Math 102

Krishanu Sankar

November 8, 2018

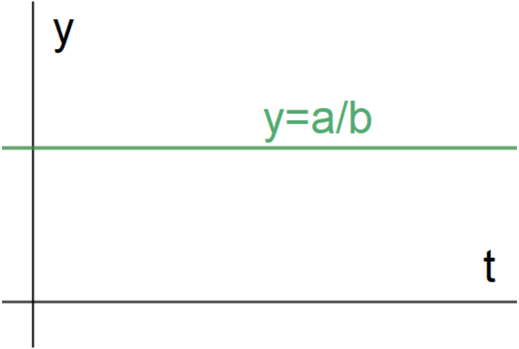
Announcements

- Reminder: You can go over your midterm with me (email for appointment)
- Resources for questions
  - Office Hours
  - Piazza
  - MLC
  - **AMS tutoring** (see ‘Getting Help’ tab on Canvas page)

Goals Today

- The general solution to the differential equation $\frac{dy}{dt} = a - by$
  - Qualitative: $b > 0$ vs. $b < 0$
  - Explicit solution using Substitution
- Applications
  - Newton's Law of Cooling
  - Drug Delivery

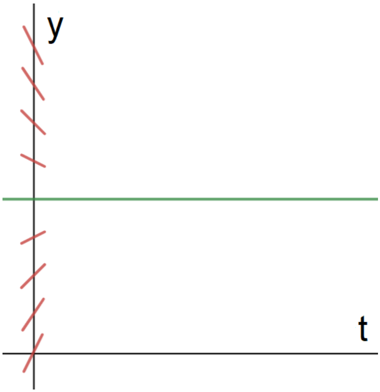
Qualitative Analysis - $\frac{dy}{dt} = a - by$

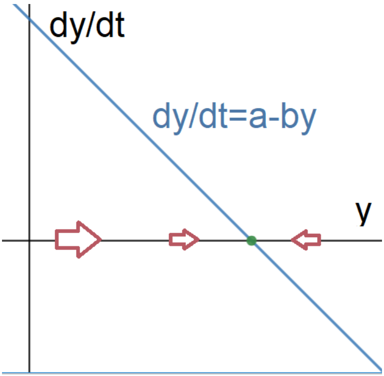


Question: Suppose that $b > 0$. Calculate $\lim_{t\to\infty} y(t)$ when

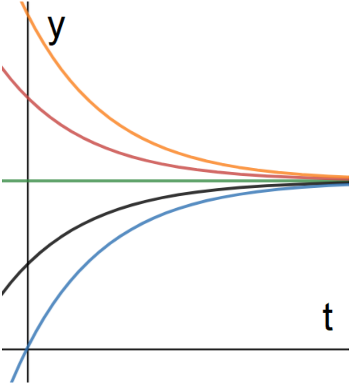
- $y(0) < a/b$
- $y(0) = a/b$
- $y(0) > a/b$

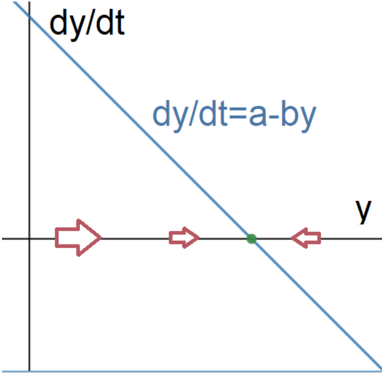
Qualitative Analysis - $\frac{dy}{dt} = a - by,\ b > 0$





Qualitative Analysis - $\frac{dy}{dt} = a - by$, $b > 0$





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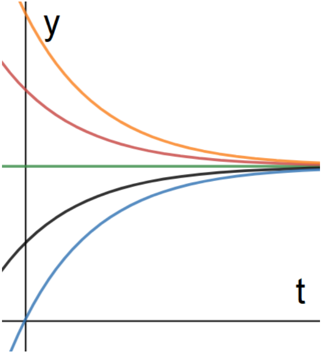
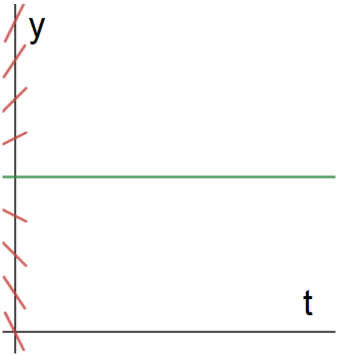
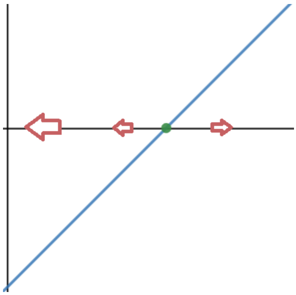


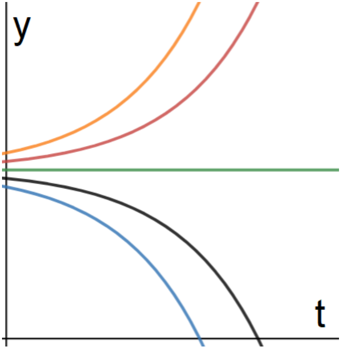
- $y(0) < a/b \implies \lim_{t\to\infty} y(t) = a/b$
- $y(0) = a/b \implies \lim_{t\to\infty} y(t) = a/b$
- $y(0) > a/b \implies \lim_{t\to\infty} y(t) = a/b$

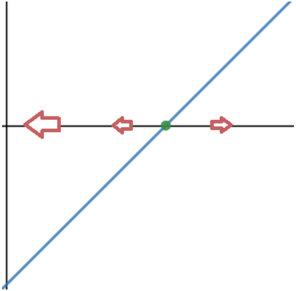
Qualitative Analysis - $\frac{dy}{dt} = a - by$, $b < 0$





Qualitative Analysis - $\frac{dy}{dt} = a - by$, $b < 0$





Solution by substitution

$$\frac{dy}{dt} = a - by = -b\left(y - \frac{a}{b}\right)$$

We make a substitution

$$z = y - \frac{a}{b} \implies \frac{dz}{dt} = \frac{dy}{dt}$$

Thus,

$$\frac{dz}{dt} = -bz \implies z(t) = Ce^{-bt}$$

Since $z(t) = y(t) - \frac{a}{b}$, this means that

$$\boxed{y(t) = \frac{a}{b} + Ce^{-bt}}$$

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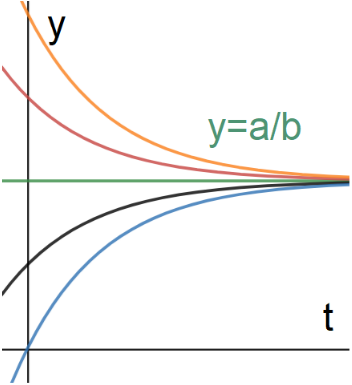
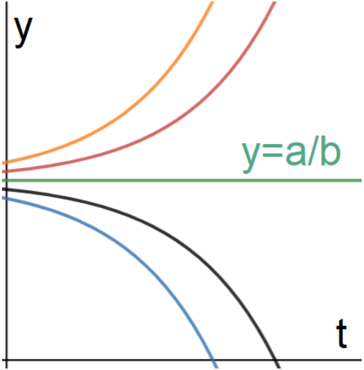


b>0: exponential decay



b<0: exponential growth

Autonomous Differential Equations

$$\frac{dy}{dt} = f(y)$$

Question: If $y(t)$ is a solution, then another solution is...

- $C \cdot y(t)$ for any constant C.
- $y(t) + C$ for any constant C.
- $y(t + C)$ for any constant C.
- Confused, help!

Autonomous Differential Equations

$$\frac{dy}{dt} = f(P)$$

Question: If $y(t)$ is a particular solution, then another solution is...

- $C \cdot y(t)$ for any constant C. (vertical dilation)
- $y(t) + C$ for any constant C. (vertical translation)
- $y(t + C)$ for any constant C (horizontal translation).

Horizontal Translation

$$\frac{dy}{dt} = a - by$$

Consider the solution $y_0(t) = \frac{a}{b} + e^{kt}$. Then

$$\begin{aligned} y_0(t+C) &= \frac{a}{b} + e^{k(t+C)} \\ &= \frac{a}{b} + e^{kC} e^{kt} \end{aligned}$$

e^{kC} can be any positive constant as C varies. Note that this doesn't give us ALL of the other solutions!

Newton's Law of Cooling: The rate of change of an object's temperature is proportional to the difference between the object's temperature and the ambient temperature.

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Question: A hot slice of pizza is placed on the countertop. The ambient temperature of the room is 20C. Let $T(t)$ denote the temperature of the pizza at time t. Which of the following is reasonable?

1. $\frac{dT}{dt} = 20 - 0.01T$
2. $\frac{dT}{dt} = 0.01(20 - T)$
3. $\frac{dT}{dt} = 0.01T - 20$
4. $\frac{dT}{dt} = 0.01(T - 20)$

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Do a sanity check: If $T < 20$, is $\frac{dT}{dt}$ positive or negative? What if $T > 20$?

Newton's Law of Cooling: The rate of change of an object's temperature is proportional to the difference between the object's temperature and the ambient temperature.

$$\frac{dT}{dt} = k(E - T)$$

where

- T is the temperature of the object.
- E is the ambient temperature. It is a constant.
- k is the proportionality constant.

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$$\frac{dT}{dt} = 0.01(20 - T)$$

Question: If $T(0) = 100$, then calculate $T(t)$.

Question: If you eat the pizza while it has a temperature higher than 60C, you will burn your mouth. How long do you have to wait before you eat the pizza?

A hot slice of pizza is placed on the countertop. The ambient temperature of the room is 20C. Let $T(t)$ denote the temperature of the pizza after t seconds.

$$\frac{dT}{dt} = 0.01(20 - T)$$

Question: If $T(0) = 100$, then calculate $T(t)$.

$$T(t) = 20 + 80e^{-0.01t}$$

Question: If you eat the pizza while it has a temperature higher than 60C, you will burn your mouth. How long do you have to wait before you eat the pizza?

$$100 \ln(2) \approx 69.3 \text{ seconds}$$

Question: A drug is delivered to a patient at a constant rate $k_{IV} > 0$. The body metabolizes the drug at a rate proportional to the amount of drug. Let $k_m > 0$ be that proportionality constant.

1. $\frac{dD}{dt} = k_{IV} - k_m D$
2. $\frac{dD}{dt} = (k_{IV} - k_m)D$
3. $\frac{dD}{dt} = k_{IV} D - k_m$
4. $\frac{dD}{dt} = -k_{IV} + k_m D$

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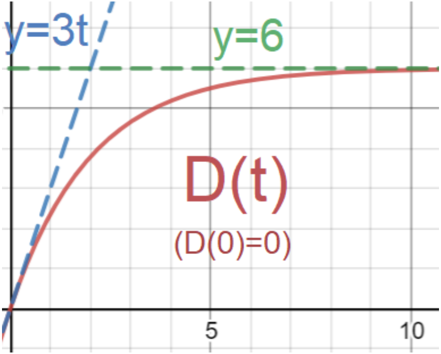
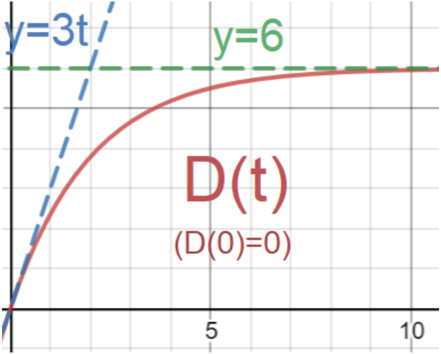


$$\frac{dD}{dt} = k_{IV} - k_m D$$

Shown is the graph of $D(t)$ given that $D(0) = 0$. The steady state is $D = 6$, and the tangent line at $t = 0$ has slope 3.

1. $k_{IV} = 1$
2. $k_{IV} = 2$
3. $k_{IV} = 3$
4. $k_{IV} = 6$



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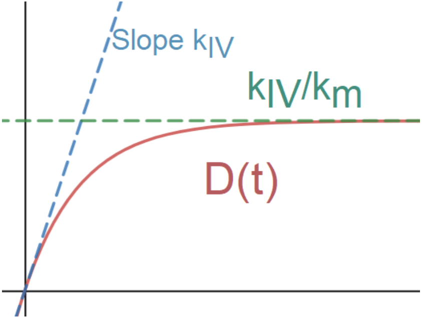
Shown is the graph of $D(t)$ given that $D(0) = 0$. The steady state is $D = 6$, and the tangent line at $t = 0$ has slope 3.

Notice that

$$D'(0) = k_{IV} - k_m D(0) = k_{IV} - k_m \cdot 0 = k_{IV}$$

The tangent line at $t = 0$ has slope $D'(0)$, by the definition of the derivative. Since this slope equals 3, we have that $D'(0) = 3$, and so $k_{IV} = 3$.

$$\frac{dD}{dt} = k_{IV} - k_m D$$



More generally, if $D(0) = 0$, then

$$D(t) = \frac{k_{IV}}{k_m} - \frac{k_{IV}}{k_m} e^{-k_m t}$$
$$= \boxed{\frac{k_{IV}}{k_m}(1 - e^{-k_m t})}$$

Question: $\frac{k_{IV}}{k_m}$ is called the **target dosage.** Calculate how long it will take for the drug dosage to reach *half* of the target dosage. What about $1/3$ of the target dosage?